CHAPTER 3

DETERMINANTS.

Reconsider the computation of the inverse of

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

"A is invertible iff $ad - bc \neq 0$"

The quantity $ad - bc$ is called det (A) and it's also denoted as det (A). It means

$$\det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Minors and Cofactors

To extend the concept of determinant to higher order square matrices, it's convenient to define $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ and then

$$\det(A) \overset{\text{def}}{=} a_{11}a_{22} - a_{12}a_{21} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

<u>Definition of det(A) for a 3x3 matrix from the definition of det(A) for a 2x2 matrix.</u>

Consider

<u>1st row</u>

$$A = \begin{bmatrix} 1 & 4 & -3 \\ -2 & -7 & 6 \\ 1 & 7 & -2 \end{bmatrix}$$

$$M_{11} = \begin{vmatrix} -7 & 6 \\ 7 & -2 \end{vmatrix} = 14 - 42 = -28.$$

$$A = \begin{bmatrix} 1 & 4 & -3 \\ -2 & -7 & 6 \\ 1 & 7 & -2 \end{bmatrix}, \quad M_{12} = \begin{vmatrix} -2 & 6 \\ 1 & -2 \end{vmatrix} = 4 - 6 = -2$$

$$A = \begin{bmatrix} 1 & 4 & -3 \\ -2 & -7 & 6 \\ 1 & 7 & -2 \end{bmatrix}, \quad M_{13} = \begin{vmatrix} -2 & -7 \\ 1 & 7 \end{vmatrix} = -14 + 7 = -7$$

Observe

$$a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13} = 1(-28) - 4(-2) - 3(-7)$$
$$= -20 + 21 = 1.$$

<u>Definitions:</u>

a) M_{11}, M_{12}, M_{13} are called <u>minors</u>

M_{11} is the minor of entry a_{11} and so forth...

b) We also define

$C_{11} = (-1)^{1+1} M_{11} = M_{11}, \quad C_{12} = (-1)^{1+2} M_{12} = -M_{12}$, and so forth.

Then, $a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} = a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13} = 1$

2'

Also, $\underline{1^{st} \text{ Column}}$

$$A = \begin{bmatrix} 1 & 4 & -3 \\ -2 & -7 & 6 \\ 1 & 7 & -2 \end{bmatrix}, \quad M_{21} = \begin{vmatrix} 4 & -3 \\ 7 & -2 \end{vmatrix} = 13$$

$$A = \begin{bmatrix} 1 & 4 & -3 \\ -2 & -7 & 6 \\ 1 & 7 & -2 \end{bmatrix}, \quad M_{31} = \begin{vmatrix} 4 & -3 \\ -7 & 6 \end{vmatrix} = 3$$

and

$$a_{11}M_{11} - a_{21}M_{21} + a_{31}M_{31} =$$
$$= 1(-28) - (-2)(13) + 1(3) = 1$$

Also,

$$M_{22} = \begin{vmatrix} 1 & -3 \\ 1 & -2 \end{vmatrix} = -2+3 = 1$$

$$M_{23} = \begin{vmatrix} 1 & 4 \\ 1 & 7 \end{vmatrix} = 7-4 = 3$$

$$M_{32} = \begin{vmatrix} 1 & -3 \\ -2 & 6 \end{vmatrix} = 6-6 = 0$$

$$M_{33} = \begin{vmatrix} 1 & 4 \\ -2 & -7 \end{vmatrix} = -7+8 = 1$$

Also, $a_{11}C_{11} + a_{21}C_{21} + a_{31}C_{31} = 1$

where $C_{i1} = (-1)^{i+1}$, $\quad i = 1,2,3.$

The Common number "1" obtained independently of the row or column considered is called

det(A).

Therefore,

$$\boxed{\det(A) = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} = 1} \quad (2.1)$$

The numbers <u>C_{11}, C_{12}, C_{13} are called cofactors</u> of their respective entries a_{11}, a_{12}, a_{13}.

But also,

$$\boxed{\det(A) = a_{11}C_{11} + a_{21}C_{21} + a_{31}C_{31} = 1} \quad (2.2)$$

The expression (2.1) is called <u>cofactor expansion of A along 1^{st} row.</u>

and (2.2) Correspond to a <u>cofactor expansion of A along 1^{st} column.</u>

Also, <u>2nd Row</u>

$$-a_{21} M_{21} + a_{22} M_{22} - a_{23} M_{23} =$$

$$= 2(13) + (-7)(1) - 6(3) = 26 - 7 - 18 = 1.$$

<u>3rd Row</u>

$$a_{31} M_{31} - a_{32} M_{32} + a_{33} M_{33} =$$

$$= 1(3) - (-7)(0) + (-2)(1) = 1$$

Similarly, the corresponding combination for columns

<u>Column 2</u>

$$-a_{12} M_{12} + a_{22} M_{22} - a_{32} M_{32} = 1$$

<u>Column 3</u>

$$a_{13} M_{13} - a_{23} M_{23} + a_{33} M_{33} = 1$$

This common number "1" will be called the

det (A), i.e., <u>$\det(A) = 1$</u>

<u>Definition.</u>- the determinants M_{ij} $(i,j = 1,2,3)$ of the 2×2 matrices that remain after the ith row and jth column are deleted from $A_{3\times3}$ are called the minor of entry a_{ij}
the number $(-1)^{i+j} M_{ij}$ is called the cofactor of entry a_{ij}

The property observed in the previous computation for this particular 3×3 matrix. is true in general for any 3×3 matrix A.

This is the content of the following theorem:

Thm.- If A is a 3×3 matrix, then regardless of which row or column of A is chosen, the number obtained by multiplying the entries in that row or column by the corresponding cofactors and adding the resulting products is always the same.

This property can be easily proved for any 2×2 matrix. In fact,

for $\underset{2\times2}{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

$M_{11} = a_{22}$, $\quad M_{12} = a_{21}$, $\quad M_{21} = a_{12}$, $\quad M_{22} = a_{11}$

$C_{11} = a_{22}$, $\quad C_{12} = -a_{21} = -M_{12}$, $\quad C_{21} = -a_{12} = -M_{21}$, $\quad C_{22} = a_{11}$

and

$$a_{11}C_{11} + a_{12}C_{12} = a_{11}a_{22} - a_{12}a_{21} \quad \text{(along } 1^{st} \text{ row)}$$

$$a_{11}C_{11} + a_{21}C_{21} = a_{11}a_{22} - a_{21}a_{12} \quad \text{(along } 1^{st} \text{ column)}.$$

Similar for 2^{nd} row and 2^{nd} column.

Therefore, the previous thm is also verified by 2×2 matrices.

Def.- If A is a 3×3 matrix, then the common number obtained by multiplying the entries in any row or column of A by the corresponding cofactors and adding the resulting products is called the determinant of A, and the sum themselves are called cofactor expansions of A. That is,

$$\det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + a_{3j}C_{3j}, \quad \text{for any column } j=1,2,3$$

(Cofactor expansion along jth column)

and

$$\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + a_{i3}C_{i3}, \quad \text{for any row } i=1,2,3$$

(Cofactor expansion along ith row)

Definition of $\det(A)$ for an $n\times n$ matrix

The previous definition about minors and cofactors can be easily extended to 4×4 and inductively to any $n\times n$ matrix. Also, the theorem remains true for the $n\times n$ matrices. Therefore the definition of $\det(A)$ for an $n\times n$ matrix is a straightforward extension of the one for 3×3 matrices given above. More specifically,

$$\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}, \quad \text{for any column } j=1,2,\ldots,n$$

$$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}, \quad \text{for any row } i=1,2,\ldots,n.$$

Computing det(A) by a Smart choice of Row or Column.

2.1) #21)

$$A = \begin{bmatrix} -3 & 0 & 7 \\ 2 & 5 & 1 \\ -1 & 0 & 5 \end{bmatrix}$$

Choose 2^{nd} column, then

$$\det(A) = 0\,C_{12} + 5\,C_{22} + 0\,C_{23} =$$

$$= 5\,M_{22} = 5\begin{vmatrix} -3 & 7 \\ -1 & 5 \end{vmatrix} =$$

$$= 5\,(-15+7) = -40.$$

Arrow technique for 3x3 matrices

$$\begin{array}{r} 72 \\ 27 \\ \hline 45 \end{array}$$

#11)

$$A = \begin{bmatrix} -2 & 1 & 4 \\ 3 & 5 & -7 \\ 1 & 6 & 2 \end{bmatrix} \begin{matrix} -2 & 1 \\ 3 & 5 \\ 1 & 6 \end{matrix} = (-20-7+72) - (20+84+6) =$$

$$= 45 - 110 = -65$$

Same result that will be obtained by applying Cofactor expansion. See proof in book.

Determinant of an Upper Triangular matrix.

$$\det\left(\begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\right) = a_{11}\underbrace{\begin{vmatrix} a_{22} & 0 \\ a_{32} & a_{33} \end{vmatrix}}_{M_{11}} = a_{11}a_{22}a_{33}$$

Inductively can be proven that $\begin{vmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & 0 \cdots & 0 \\ \vdots & & \ddots & 0 \\ a_{n1} & \cdots & \cdots & a_{nn} \end{vmatrix} = a_{11}a_{22}\cdots a_{nn}.$

Consider

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 2 & 0 \\ 4 & 5 & 6 \end{bmatrix}$$

then,

$$\det A = 1 \begin{vmatrix} 2 & 0 \\ 5 & 6 \end{vmatrix} - 0 \begin{vmatrix} 3 & 0 \\ 4 & 6 \end{vmatrix} + 0 \begin{vmatrix} 3 & 2 \\ 4 & 5 \end{vmatrix} = 1 \times 2 \times 6 = 12.$$

This result is true for any matrix $A_{n \times n}$.

Thm. 2.- If

$$A = \begin{bmatrix} a_{11} & 0 & \dots & 0 \\ a_{21} & a_{22} & \dots & 0 \\ \vdots & & \ddots & 0 \\ a_{n1} & \dots & & a_{nn} \end{bmatrix}$$

is a "triangular matrix"

then,

$$\det A = a_{11} a_{22} \dots a_{nn}.$$

Show in class

$n=2$ ✓

$n=k$ ✓

$$\begin{vmatrix} a_{11} & \cdots & \\ & & a_{kk} \end{vmatrix} = a_{..} \ldots a$$

$n=k+1$

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1,k+1} \\ a_{21} & a_{22} & 0 & a_{2,k+1} \\ \vdots & & & \vdots \\ a_{k1} & a_{k2} & \cdots a_{kk} & a_{k,k+1} \\ a_{k+1,1} & \cdots & & a_{k+1,k+1} \end{vmatrix}_{(k+1)\times(k+1)} = a_{k+1,k+1} \underbrace{\begin{vmatrix} a_{11} & 0 & \cdots & 0 & 0 \\ a_{21} & a_{22} & 0 & \cdots & 0 \\ \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kk} \end{vmatrix}}_{= M_{k,k}} =$$

induct. ✓

$$= a_{k+1,k+1}(a_{11} \ldots a_{kk}).$$

Show that $\det(A) = \begin{vmatrix} a_{11} & & \\ a_{21} & \ddots & 0 \\ \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix} = a_{11} \ldots a_{nn}$

By induction

It's true for matrices $A_{2\times 2}$ $(n=2)$ for a $A_{1\times 1}$ $A=[a_{11}]$ $\det(A) \equiv a_{11}$

Since $\begin{vmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} + 0\,a_{21}$

Assume true for $n=k$, so $\begin{vmatrix} a_{11} & 0 & \cdots & 0 \\ \vdots & \ddots \\ a_{k1} & & & a_{kk} \end{vmatrix} = a_{11} \ldots a_{kk}$

then for $n=k+1$, $\begin{vmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & 0 \\ \vdots \\ a_{k1} & \cdots & a_{kk} & 0 \\ a_{k+1,1} & & & a_{k+1,k+1} \end{vmatrix} = a_{k+1,k+1}(-1)^{2k+2} \begin{vmatrix} a_{11} & 0 & \cdots & 0 \\ \vdots & \ddots \\ a_{k1} & & a_{kk} \end{vmatrix}$ induct Hip. $= a_{k+1,k+1}(a_{11} \ldots a_{kk})$

$= a_{k+1,k+1} \ldots a_{11}.$